GENERATION OF MUSICAL PATTERNS THROUGH OPERADS

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RÉSUMÉ

Nous introduisons la notion de multi-motif, une abstraction combinatoire des phrases musicales à plusieurs voix. L'intérêt de cette approche réside dans le fait qu'il devient possible de composer deux multi-motifs pour en produire un plus long. Ceci s'inscrit dans un contexte algébrique puisque l'ensemble des multi-motifs possède une structure dite d'opérade ; les opérades étant des structures offrant une formalisation de la notion d'opérateur et de leurs compositions. Cette vision des phrases musicales comme des opérateurs permet de réaliser ainsi des calculs sur ces dernières et admet des applications en musique générative : étant donné un ensemble de courts motifs, nous proposons divers algorithmes pour produire de manière aléatoire une nouvelle phrase plus longue inspirée des motifs initiaux.

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We introduce the notion of multi-pattern, a combinatorial abstraction of polyphonic musical phrases. The interest of this approach lies in the fact that this offers a way to compose two multi-patterns in order to produce a longer one. This dives musical phrases into an algebraic context since the set of multi-patterns has the structure of an operad ; operads being structures offering a formalization of the notion of operators and their compositions. Seeing musical phrases as operators allows us to perform computations on phrases and admits applications in generative music : given a set of short patterns, we propose various algorithms to randomly generate a new and longer phrase inspired by the inputted patterns.

1. INTRODUCTION

Generative music is a subfield of computational musicology in which the focus lies on the automatic creation of musical material. This creation is based on algorithms accepting inputs to influence the result obtained, and having a randomized behavior in the sense that two executions of the algorithm with the same inputs produce different results. Several very different approaches exist. For instance, some of them use Markov chains, others genetic algorithms [10], still others neural networks [2], or even formal grammars [7,8]. The way in which such algorithms represent and manipulate musical data is crucial. Indeed, the data structures used to represent musical phrases orient the nature of the operations we can define of them. Considering operations producing new phrases from old ones is important to specify algorithms to randomly generate music. A possible way for this purpose consists to give as input some musical phrases and the algorithm creates a new one by blending them through operations. Therefore, the willingness to endow the infinite set of all musical phrases with operations in order to obtain suitable algebraic structures is a promising approach. Such interactions between music and algebra is a fruitful field of investigation [1, 9].

In this work, we propose to use tools coming from combinatorics and algebraic combinatorics to represent musical phrases and operations on them, in order to introduce generative music algorithms close to the family of those based on formal grammars. More precisely, we introduce the music box model, a very simple model to represent polyphonic phrases, called multi-patterns. The infinite set of all these objects admits the structure of an operad. Such structures originate from algebraic topology and are used nowadays also in algebraic combinatorics and in computer science [4, 11]. Roughly speaking, in these algebraic structures, the elements are operations with several inputs and the composition law is the usual composition of operators. Since the set of multi-patterns forms an operad, one can regard each pattern as an operation. The fallout of this is that each pattern is, at the same time, a musical phrase and an operation acting on musical phrases. In this way, our music box model and its associated operad provide an algebraic and combinatorial framework to perform computations on musical phrases.

All this admits direct applications to design random generation algorithms since, as introduced by the author in [5], given an operad there exist algorithms to generate some of its elements. These algorithms are based upon bud generating systems, which are general formal grammars based on colored operads [12]. In the present work, we propose three different variations of these algorithms to produce new musical phrases from old ones. More precisely, our algorithm works as follows. It takes as input a finite set of multi-patterns and an integer value to influence the size of the output. It works iteratively by choosing patterns from the initial collection in order to alter the current one by performing a composition using the operad structure. As we shall explain, the initial patterns can be colored in order to forbid some compositions and avoid in this way some musical intervals for instance. These generation algorithms are not intended to write complete musical pieces ; they are for obtaining, from short old patterns,

a similar but longer one, presenting possibly new ideas to the human composer.

This text is organized as follows. Section 2 is devoted to setting our context and notations about music theory and to introduce the music box model. In Section 3, we begin by presenting a brief overview of operad theory and we build step by step the music box operad. For this, we introduce first an operad on sequences of scale degrees, an operad on rhythm patterns, and then an operad on monophonic patterns to end with the operad of multi-patterns. Three random generation algorithms for multi-patterns are introduced in Section 4. Finally, Section 5 provides some concrete applications of the previous algorithms. We focus here on random variations of a monophonic musical phrase as input leading to random changes of rhythm, harmonizations, and arpeggiations.

In this version of this work, most of the proofs of the announced results are omitted due to lack of space. A computer implementation of all the presented algorithms is, as well as its source code and *concrete* examples, available at [6].

In the remainder of this article, we will use the following notations and conventions. For any integer n, $[n]$ denotes the set $\{1, \dots, n\}$. If a is a letter and n is a nonnegative integer, a^n is the word consisting in n occurrences of a. In particular, a^0 is the empty word ϵ.

2. THE MUSIC BOX MODEL

The purpose of this section is to set some definitions and some conventions about music theory, and introduce multi-patterns that are abstractions of musical phrases.

2.1. Notes and scales

We fit into the context of an η tone equal temperament, also written as η-TET, where η is any nonnegative integer. An η-*note* is a pair (k, n) where $0 \leqslant k \leqslant \eta - 1$ and $n \in \mathbb{Z}$. We shall write k_n instead of (k, n). The integer n is the *octave index* and k is the *step index* of k_n. The set of all η-notes is denoted by $\mathcal{N}^{(\eta)}$. Despite this level of generality, and even if all the concepts developed in the sequel work for any η, in most applications and examples we shall consider that $\eta = 12$. Therefore, under this convention, we simply call *note* any 12-*note* and write $\mathcal{N}$ for $\mathcal{N}^{(12)}$. We set in this context of 12-TET the "middle C" as the note 0_4, which is the first step of the octave of index 4. An η-*scale* is an integer composition $\boldsymbol{\lambda}$ of η, that is a sequence $(\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_\ell)$ of nonnegative integers satisfying $\boldsymbol{\lambda}_1 + \dots + \boldsymbol{\lambda}_\ell = \eta$. The *length* of $\boldsymbol{\lambda}$ is the number $\ell(\boldsymbol{\lambda}) := \ell$ of its elements. We simply call *scale* any 12-scale. For instance, $(2, 2, 1, 2, 2, 2, 1)$ is the major natural scale, $(2, 1, 2, 2, 1, 3, 1)$ is the harmonic minor scale, and $(2, 1, 4, 1, 4)$ is the Hirajoshi scale. This encoding of a scale by an integer composition is also known under the terminology of interval pattern.

A *rooted scale* is a pair $(\boldsymbol{\lambda}, r)$ where $\boldsymbol{\lambda}$ is a scale and r is a note. This rooted scale describes a subset $\mathcal{N}_{(\boldsymbol{\lambda}, r)}$ of $\mathcal{N}$ consisting in the notes reachable from r by following the steps prescribed by the values $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \dots, \boldsymbol{\lambda}_{\ell(\boldsymbol{\lambda})}$ of $\boldsymbol{\lambda}$. For instance, if $\boldsymbol{\lambda}$ is the Hirajoshi scale, then

$$\mathcal{N}_{(\boldsymbol{\lambda}, 0_4)} = \{\dots, 7_3, 8_3, \mathbf{0_4}, 2_4, 3_4, 7_4, 8_4, 0_5, \dots\} \quad (1)$$

If $\boldsymbol{\lambda}$ is the major natural scale, then

$$\mathcal{N}_{(\boldsymbol{\lambda}, 2_4)} = \{\dots, 1_4, \mathbf{2_4}, 4_4, 6_4, 7_4, 9_4, 11_4, 1_5, 2_5, \dots\} \quad (2)$$

2.2. Patterns

We now introduce degree patterns, rhythm patterns, patterns, and finally multi-patterns.

A *degree d* is any element of $\mathbb{Z}$. Negative degrees are denoted by putting a bar above their absolute value. For instance, -3 is denoted by $\bar{3}$. A *degree pattern* $\mathbf{d}$ is a finite word $\mathbf{d}_1 \dots \mathbf{d}_\ell$ of degrees. The *arity* of $\mathbf{d}$, also denoted by $|\mathbf{d}|$, is the number ℓ of its elements.

Given a rooted scale $(\boldsymbol{\lambda}, r)$, a degree pattern $\mathbf{d}$ specifies a sequence of notes by assigning to the degree 0 the note r, to the degree 1 the following higher note in $\mathcal{N}_{(\boldsymbol{\lambda}, r)}$ next to r, to the degree $\bar{1}$ the lower note in $\mathcal{N}_{(\boldsymbol{\lambda}, r)}$ next to r, and so on. For instance, the degree pattern $10\bar{2}3507$ specifies, in the context of the rooted scale $(\boldsymbol{\lambda}, 0_4)$ where $\boldsymbol{\lambda}$ is the major natural scale, the sequence of notes

$$2_4, 0_4, 9_3, 7_3, 9_4, 0_4, 0_5 \quad (3)$$

A *rhythm pattern* $\mathbf{r}$ is a finite word $\mathbf{r}_1 \dots \mathbf{r}_\ell$ on the alphabet $\{\square, \blacksquare\}$. The symbol $\square$ is a *rest* and the symbol $\blacksquare$ is a *beat*. The *length* of $\mathbf{r}$ is ℓ and the *arity* $|\mathbf{r}|$ of $\mathbf{r}$ is its number of occurrences of beats. The *duration sequence* of a rhythm pattern $\mathbf{r}$ is the unique sequence $(\alpha_0, \alpha_1, \dots, \alpha_{|\mathbf{r}|})$ of nonnegative integers such that

$$\mathbf{r} = \square^{\alpha_0} \; \blacksquare \; \square^{\alpha_1} \; \cdots \; \blacksquare \; \square^{\alpha_{|\mathbf{r}|}} \quad (4)$$

The rhythm pattern $\mathbf{r}$ specifies a rhythm wherein each beat has a relative duration : the rhythm begins with a silence of α_0 units of time, followed by a first beat sustained $1 + \alpha_1$ units of time, and so on, and finishing by a last beat sustained $1 + \alpha_{|\mathbf{r}|}$ units of time. We adopt here the convention that each rest and beat last each the same amount of time of one eighth of the duration of a whole note. Therefore, given a tempo specifying how many there are rests and beats by minute, any rhythm pattern encodes a rhythm.

For instance, let us consider the rhythm pattern

$$\mathbf{r} := \square\blacksquare\blacksquare\square\blacksquare\square\square\square\blacksquare\blacksquare\square\blacksquare \quad (5)$$

The duration sequence of $\mathbf{r}$ is $(1, 0, 1, 3, 0, 1, 0)$ so that $\mathbf{r}$ specifies the rhythm consisting in an eighth rest, an eighth note, a quarter note, a half note, an eighth note, a quarter note, and finally an eighth note.

A *pattern* is a pair $\mathbf{p} := (\mathbf{d}, \mathbf{r})$ such that $|\mathbf{d}| = |\mathbf{r}|$. The *arity* $|\mathbf{p}|$ of $\mathbf{p}$ is the arity of both $\mathbf{d}$ and $\mathbf{r}$, and the *length* $\ell(\mathbf{p})$ of $\mathbf{d}$ is the length $\ell(\mathbf{r})$ of $\mathbf{r}$.

In order to handle concise notations, we shall write any pattern $(\mathbf{d}, \mathbf{r})$ as a word $\mathbf{p}$ on the alphabet $\{\square\} \cup \mathbb{Z}$ where the subword of $\mathbf{p}$ obtained by removing all occurrences

of $\square$ is the degree pattern $\mathbf{d}$, and the word obtained by replacing in $\mathbf{p}$ each integer by $\blacksquare$ is the rhythm pattern $\mathbf{r}$. For instance,

$$1\square\square\bar{2}\square12 \qquad (6)$$

is the concise notation for the pattern

$$(1\bar{2}12, \blacksquare\square\square\blacksquare\square\blacksquare\blacksquare) \qquad (7)$$

For this reason, thereafter, we shall see and treat any pattern $\mathbf{p}$ as a finite word $\mathbf{p}_1 \dots \mathbf{p}_\ell$ on the alphabet $\{\square\} \cup \mathbb{Z}$. Remark that the length of $\mathbf{p}$ is ℓ and that its arity is the number of letters of $\mathbb{Z}$ it has.

Given a rooted scale $(\boldsymbol{\lambda}, r)$ and a tempo, a pattern $\mathbf{p} := (\mathbf{d}, \mathbf{r})$ specifies a musical phrase, that is a sequence of notes arranged into a rhythm. The notes of the musical phrase are the ones specified by the degree pattern $\mathbf{d}$ and their relative durations are specified by the rhythm pattern $\mathbf{r}$. For instance, consider the pattern

$$\mathbf{d} := 0\square12\bar{1}\square01\bar{2}\square\bar{1}00\square\square\square \qquad (8)$$

By choosing the rooted scale $(\boldsymbol{\lambda}, 9_3)$ where $\boldsymbol{\lambda}$ is the harmonic minor scale, and by setting 128 as tempo, one obtains the musical phrase

For any positive integer m, an m-*multi-pattern* is an m-tuple $\mathbf{m} := \left(\mathbf{m}^{(1)}, \dots, \mathbf{m}^{(m)}\right)$ of patterns such that all $\mathbf{m}^{(i)}$ have the same arity and the same length. The *arity* $|\mathbf{m}|$ of $\mathbf{m}$ is the common arity of all the $\mathbf{m}^{(i)}$, and the *length* $\ell(\mathbf{m})$ of $\mathbf{m}$ is the common length of all the $\mathbf{m}^{(i)}$. An m-multi-pattern $\mathbf{m}$ is denoted through a matrix of dimension $m \times \ell(\mathbf{m})$, where the i-th row contains the pattern $\mathbf{m}^{(i)}$ for any $i \in [m]$. For instance,

$$\mathbf{m} := \begin{bmatrix} 0 & \square & 1 & \square & 1 \\ \square & \bar{2} & 3 & \square & 0 \end{bmatrix} \qquad (9)$$

is a 2-multi-pattern having arity 3 and length 5. The fact that all patterns of an m-multi-pattern must have the same length ensures that they last the same amount of units of time. This is important since an m-multi-pattern is used to handle musical sequences consisting in m stacked voices. The condition about the arities of the patterns, and hence, about the number of degrees appearing in these, is a particularity of our model and comes from algebraic reasons. This will be clarified later in this article.

Given a rooted scale $(\boldsymbol{\lambda}, r)$ and a tempo, an m-multi-pattern $\mathbf{m}$ specifies a musical phrase obtained by considering the musical phrases specified by each $\mathbf{m}^{(i)}$, $i \in [m]$, each forming a voice. For instance, consider the 2-multi-pattern

$$\mathbf{m} := \begin{bmatrix} 0 & 4 & \square & 4 & 0 & 0 \\ \bar{7} & \bar{7} & 0 & \square & \bar{3} & \bar{3} \end{bmatrix} \qquad (10)$$

By choosing the rooted scale $(\boldsymbol{\lambda}, 9_3)$ where $\boldsymbol{\lambda}$ is the minor natural scale and by setting 128 as tempo, one obtains the musical phrase

Due to the fact that m-multi-patterns evoke paper tapes of a programmable music box, we call *music box model* the model just described to represent musical phrases by m-multi-patterns within the context of a rooted scale and a tempo.

3. OPERAD STRUCTURES

The purpose of this section is to introduce an operad structure on multi-patterns, called music box operad. The main interest of endowing the set of multi-patterns with the structure of an operad is that this leads to an algebraic framework to perform computations on patterns.

3.1. A primer on operads

We set here the elementary notions of operad theory used in the sequel. Most of them come from [4].

A *graded set* is a set $\mathcal{O}$ decomposing as a disjoint union

$$\mathcal{O} := \bigsqcup_{n \in \mathbb{N}} \mathcal{O}(n) \qquad (11)$$

where the $\mathcal{O}(n)$, $n \in \mathbb{N}$, are sets. For any $x \in \mathcal{O}$, there is by definition a unique $n \in \mathbb{N}$ such that $x \in \mathcal{O}(n)$ called *arity* of x and denoted by $|x|$.

A *nonsymmetric operad*, or an *operad* for short, is a triple $(\mathcal{O}, \circ_i, \mathbf{1})$ such that $\mathcal{O}$ is a graded set, $\circ_i$ is a map

$$\underset{i}{\circ} : \mathcal{O}(n) \times \mathcal{O}(m) \to \mathcal{O}(n + m - 1), \quad i \in [n] \qquad (12)$$

called *partial composition* map, and $\mathbf{1}$ is a distinguished element of $\mathcal{O}(1)$, called *unit*. This data has to satisfy, for any $x, y, z \in \mathcal{O}$, the three relations

$$\left(x \underset{i}{\circ} y\right) \underset{i+j-1}{\circ} z = x \underset{i}{\circ} \left(y \underset{j}{\circ} z\right), \; i \in [\|x\|], j \in [\|y\|] \qquad (13)$$

$$\left(x \underset{i}{\circ} y\right) \underset{j+|y|-1}{\circ} z = \left(x \underset{j}{\circ} z\right) \underset{i}{\circ} y, \; 1 \leqslant i < j \leqslant |x| \qquad (14)$$

$$\mathbf{1} \underset{1}{\circ} x = x = x \underset{i}{\circ} \mathbf{1}, \; i \in [\|x\|] \qquad (15)$$

Intuitively, an operad is an algebraic structure wherein each element can be seen as an operator having $|x|$ inputs and one output. Such an operator is depicted as

where the inputs are at the bottom and the output at the top. Given two operations x and y of $\mathcal{O}$, the partial composition $x \circ_i y$ is a new operator obtained by composing y

into x onto its i-th input. Pictorially, this partial composition expresses as

$$\text{(17)}$$

Relations (13), (14), and (15) become clear when they are interpreted into this context of abstract operators and rooted trees.

Let $(\mathcal{O}, \circ_i, \mathbf{1})$ be an operad. The *full composition* map of $\mathcal{O}$ is the map

$$\circ : \mathcal{O}(n) \times \mathcal{O}(m_1) \times \cdots \times \mathcal{O}(m_n) \to \mathcal{O}(m_1 + \cdots + m_n) \tag{18}$$

defined, for any $x \in \mathcal{O}(n)$ and $y_1, \ldots, y_n \in \mathcal{O}$ by

$$x \circ [y_1, \ldots, y_n] := \left(\ldots \left(\left(x \underset{n}{\circ} y_n \right) \underset{n-1}{\circ} y_{n-1} \right) \ldots \right) \underset{1}{\circ} y_1 \tag{19}$$

Intuitively, $x \circ [y_1, \ldots, y_n]$ is obtained by simultaneously grafting the outputs of the y_i onto the i-th inputs of x.

Let $(\mathcal{O}', \circ_i', \mathbf{1}')$ be a second operad. A map $\phi : \mathcal{O} \to \mathcal{O}'$ is an *operad morphism* if for any $x \in \mathcal{O}(n)$, $\phi(x) \in \mathcal{O}'(n)$, $\phi(\mathbf{1}) = \mathbf{1}'$, and for any $x, y \in \mathcal{O}$ and $i \in [|x|]$,

$$\phi(x \underset{i}{\circ} y) = \phi(x) \underset{i}{\circ}' \phi(y) \tag{20}$$

If instead (20) holds by replacing the second occurrence of i by $|x|+1-i$, then ϕ is an *operad antimorphism*. We say that $\mathcal{O}'$ is a *suboperad* of $\mathcal{O}$ if for any $n \in \mathbb{N}$, $\mathcal{O}'(n)$ is a subset of $\mathcal{O}(n)$, $\mathbf{1} = \mathbf{1}'$, and for any $x, y \in \mathcal{O}'$ and $i \in [|x|]$, $x \circ_i y = x \circ_i' y$. For any subset $\mathfrak{G}$ of $\mathcal{O}$, the *operad generated* by $\mathfrak{G}$ is the smallest suboperad $\mathcal{O}^{\mathfrak{G}}$ of $\mathcal{O}$ containing $\mathfrak{G}$. When $\mathcal{O}^{\mathfrak{G}} = \mathcal{O}$ and $\mathfrak{G}$ is minimal with respect to the inclusion among the subsets of $\mathfrak{G}$ satisfying this property, $\mathfrak{G}$ is a *minimal generating set* of $\mathcal{O}$ and its elements are *generators* of $\mathcal{O}$.

The *Hadamard product* of $\mathcal{O}$ and $\mathcal{O}'$ is the operad $\mathcal{O} \boxtimes \mathcal{O}'$ defined, for any $n \in \mathbb{N}$, by $(\mathcal{O} \boxtimes \mathcal{O}')(n) := \mathcal{O}(n) \times \mathcal{O}'(n)$, endowed with the partial composition map $\circ_i''$ defined, for any $(x, x'), (y, y') \in \mathcal{O} \boxtimes \mathcal{O}'$ and $i \in [|(x, x')|]$, by

$$(x, x') \underset{i}{\circ}'' (y, y') := \left(x \underset{i}{\circ} y, x' \underset{i}{\circ}' y' \right) \tag{21}$$

and having $(\mathbf{1}, \mathbf{1}')$ as unit.

3.2. The music box operad

We build an operad on multi-patterns step by step by introducing an operad on degree patterns and an operad on rhythm patterns. The operad of patterns is constructed as the Hadamard product of the two previous ones. Finally, the operad of multi-patterns if a suboperad of an iterated Hadamard product of the operad of patterns with itself.

Let DP be the graded collection of all degree patterns, wherein for any $n \in \mathbb{N}$, DP(n) is the set of all degree patterns of arity n. Let us define on DP the partial composition $\circ_i$ wherein, for any degree patterns $\mathbf{d}$ and $\mathbf{d}'$, and any integer $i \in [|\mathbf{d}|]$,

$$\mathbf{d} \underset{i}{\circ} \mathbf{d}' := \mathbf{d}_1 \ldots \mathbf{d}_{i-1} \left(\mathbf{d}_i + \mathbf{d}'_1 \right) \ldots$$
$$\ldots \left(\mathbf{d}_i + \mathbf{d}'_{|\mathbf{d}'|} \right) \mathbf{d}_{i+1} \ldots \mathbf{d}_{|\mathbf{d}|} \tag{22}$$

For instance,

$$0\mathbf{1}234 \underset{2}{\circ} \bar{\mathbf{1}}\mathbf{10} = 00\mathbf{21}234 \tag{23}$$

We denote by ϵ the empty degree pattern. This element is the only one of DP(0).

Proposition 3.2.1. *The triple* $(\mathsf{DP}, \circ_i, 0)$ *is an operad.*

Démonstration. This is the consequence of the fact that $(\mathsf{DP}, \circ_i, 0)$ is the image of the monoid $(\mathbb{Z}, +, 0)$ by the construction T defined in [3]. Since this construction associates an operad with any monoid, the result follows. $\square$

We call DP the *degree pattern operad*.

Proposition 3.2.2. *The operad* DP *admits* $\{\epsilon, \bar{1}, 1, 00\}$ *and* $\{\epsilon, \bar{1}1\}$ *as minimal generating sets.*

Let RP be the graded collection of all rhythm patterns, wherein for any $n \in \mathbb{N}$, RP(n) is the set of all rhythm patterns of arity n. Let us define on RP the partial composition $\circ_i$ wherein, for any rhythm patterns $\mathbf{r}$ and $\mathbf{r}'$, and any integer $i \in [|\mathbf{r}|]$, $\mathbf{r} \circ_i \mathbf{r}'$ is obtained by replacing the i-th occurrence of $\blacksquare$ in $\mathbf{r}$ by $\mathbf{r}'$. For instance,

$$\blacksquare\blacksquare\square\blacksquare\square\square\blacksquare \underset{3}{\circ} \square\blacksquare\square\blacksquare = \blacksquare\blacksquare\square\ \square\blacksquare\square\blacksquare\ \square\square\blacksquare \tag{24}$$

We denote by ϵ the empty rhythm pattern. This element is not the only one of RP(0) since RP$(0) = \{\square^{\alpha} : \alpha \in \mathbb{N}\}$.

Proposition 3.2.3. *The triple* $(\mathsf{RP}, \circ_i, \blacksquare)$ *is an operad*

We call RP the *rhythm pattern operad*.

Proposition 3.2.4. *The operad* RP *admits* $\{\epsilon, \square, \blacksquare\blacksquare\}$ *as minimal generating set.*

Let P be the operad defined as

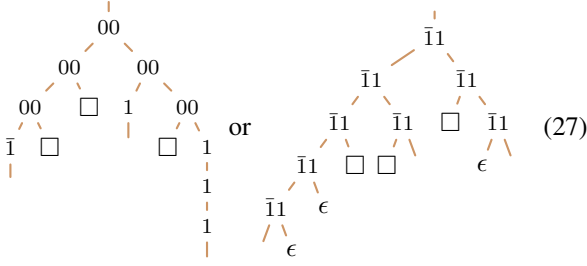$$\mathsf{P} := \mathsf{DP} \boxtimes \mathsf{RP} \tag{25}$$

Since a pattern is a pair $(\mathbf{d}, \mathbf{r})$ where $\mathbf{d}$ is a degree pattern and $\mathbf{r}$ is a rhythm pattern of the same arity, for any $n \in \mathbb{N}$, P(n) is in fact the set of all patterns of arity n. For this reason, P is the graded set of all patterns. We call P the *pattern operad*. For instance, by using the concise notation for patterns,

$$\square\bar{2}1\square1 \underset{2}{\circ} 0\square\bar{1} = \square\bar{2}\ \mathbf{1}\square\mathbf{0}\ \square\mathbf{1} \tag{26}$$

We denote by ϵ the empty pattern.

Proposition 3.2.5. *The operad* P *admits* $\{\epsilon, \square, \bar{1}, 1, 00\}$ *and* $\{\epsilon, \square, \bar{1}1\}$ *as minimal generating sets.*

As a consequence of proposition 3.2.5, any pattern **p** expresses as a tree on the internal nodes in $\{\epsilon, \square, \bar{1}, 1, 00\}$ or in $\{\epsilon, \square, \bar{1}1\}$. For instance, the pattern $\mathbf{p} := \bar{1}\square\square 1\square 3$ expresses as the trees

$$
\begin{array}{cc}
\begin{array}{c}
00 \\
00 \quad 00 \\
00 \quad \square \quad 1 \quad 00 \\
\bar{1} \quad \square \qquad \square \quad 1 \\
\square \qquad\qquad 1 \\
\qquad\qquad 1
\end{array}
& \text{or} &
\begin{array}{c}
\bar{1}1 \\
\bar{1}1 \quad\quad \bar{1}1 \\
\bar{1}1 \quad \bar{1}1 \quad \square \quad \bar{1}1 \\
\bar{1}1 \quad \square \quad \square \qquad \epsilon \\
\bar{1}1 \quad \epsilon \\
\epsilon
\end{array}
& (27)
\end{array}
$$

respectively for the two previous generating sets.

For any positive integer m, let P'_m be operad defined through the iterated Hadamard product

$$
\mathsf{P}'_m := \underbrace{\mathsf{P} \boxtimes \cdots \boxtimes \mathsf{P}}_{m \text{ terms}} \qquad (28)
$$

Let also P_m be the subset of P'_m restrained on the m-tuples $\left(\mathbf{m}^{(1)}, \dots, \mathbf{m}^{(m)}\right)$ such that

$$
\ell\left(\mathbf{m}^{(1)}\right) = \cdots = \ell\left(\mathbf{m}^{(m)}\right). \qquad (29)
$$

Theorem 3.2.6. *For any positive integer m, P_m is an operad.*

Since an m-multi-pattern $\left(\mathbf{m}^{(1)}, \dots, \mathbf{m}^{(m)}\right)$ is an m-tuple where all $\mathbf{m}^{(i)}$ have the same arity and the same length, for any $m \in \mathbb{N}$, P_m is the graded set of all m-multi-patterns. By Theorem 3.2.6, P_m is an operad, called m-*music box operad*.

By using the matrix notation for m-multi-patterns, we have for instance respectively in P_2 and in P_3,

$$
\begin{bmatrix} \square & \bar{2} & \bar{1} & \square & 0 \\ 0 & 1 & \square & \square & 1 \end{bmatrix} \circ_2 \begin{bmatrix} \mathbf{1} & \square & \mathbf{0} & \mathbf{0} \\ \mathbf{3} & \square & \mathbf{0} & \mathbf{4} \end{bmatrix} = \begin{bmatrix} \square & \bar{2} & \mathbf{0} & \square & \mathbf{0} & \bar{\mathbf{0}} & \square & 0 \\ 0 & \bar{2} & \square & \mathbf{1} & 5 & \square & \square & 1 \end{bmatrix} \quad (30)
$$

This definition of the m-music box operad P_m explains why all the patterns of an m-multi-pattern must have the same arity. This is a consequence of the general definition of the Hadamard product of operads.

For any sequence $(\alpha_1, \dots, \alpha_m)$ of integers of $\mathbb{Z}$ and $\beta \in \mathbb{N}$, let

$$
\phi_{(\alpha_1, \dots, \alpha_m), \beta} : \mathsf{P}_m \to \mathsf{P}_m \qquad (31)
$$

be the map such that, for any $\mathbf{m} := \left(\mathbf{m}^{(1)}, \dots, \mathbf{m}^{(m)}\right) \in \mathsf{P}_m$, $\phi_{(\alpha_1, \dots, \alpha_m), \beta}(\mathbf{m})$ is the m-multi-pattern obtained by multiplying each degree of $\mathbf{m}^{(j)}$ by α_j and by replacing each occurrence of $\square$ in $\mathbf{m}$ by β occurrences of $\square$. For instance,

$$
\phi_{(2,0,-1),2}\left(\begin{bmatrix} 1 & \square & \square & 2 \\ \square & 1 & \square & 3 \\ 3 & 1 & \square & \square \end{bmatrix}\right) = \begin{bmatrix} 2 & \square & \square & \square & \square & 4 \\ \square & \square & 0 & \square & \square & 0 \\ \bar{3} & \bar{1} & \square & \square & \square & \square \end{bmatrix} \quad (32)
$$

Proposition 3.2.7. *For any positive integer m, any sequence $(\alpha_1, \dots, \alpha_m)$ of integers, and any nonnegative integer β, the map $\phi_{(\alpha_1, \dots, \alpha_m), \beta}$ is an operad endomorphism of P_m.*

Let also the map $\mathrm{mir} : \mathsf{P}_m \to \mathsf{P}_m$ be the map such that, for any $\mathbf{m} \in \mathsf{P}_m$, $\mathrm{mir}(\mathbf{m})$ is the m-multi-pattern obtained by reading the $\mathbf{m}$ from right to left. For instance,

$$
\mathrm{mir}\left(\begin{bmatrix} 1 & \square & \square & 2 \\ \square & 1 & \square & 3 \\ 3 & 1 & \square & \square \end{bmatrix}\right) = \begin{bmatrix} 2 & \square & \square & 1 \\ 3 & \square & 1 & \square \\ \square & \square & 1 & 3 \end{bmatrix} \quad (33)
$$

Proposition 3.2.8. *For any positive integer m, the map* mir *sending any m-multi-pattern to its mirror is an operad anti-automorphism of P_m.*

Due to the m-music box operad and more precisely, to the operad structure on m-multi-patterns, we can see any m-multi-pattern as an operator. Therefore, we can build m-multi-patterns and then musical sequences by considering some compositions of small building blocks m-multi-patterns. For instance, by considering the small 2-multi-patterns

$$
\mathbf{m}_1 := \begin{bmatrix} 0 & \square \\ \square & 0 \end{bmatrix}, \quad \mathbf{m}_2 := \begin{bmatrix} 1 & 0 & 1 \\ \bar{7} & 0 & 0 \end{bmatrix}, \quad \mathbf{m}_3 := \begin{bmatrix} 1 & 2 & \square & 3 \\ \bar{1} & 0 & \square & 1 \end{bmatrix} \quad (34)
$$

one can build a new 2-multi-pattern by composing them as specified by the tree

$$
\begin{array}{c}
\mathbf{m}_2 \\
\mathbf{m}_1 \qquad \mathbf{m}_3 \\
\mathbf{m}_2
\end{array} \qquad (35)
$$

This produces the new 2-multi-pattern

$$
\begin{bmatrix} 1 & 1 & 0 & 1 & \square & 2 & 3 & \square & 3 \\ \bar{7} & \square & \bar{7} & 0 & 0 & \bar{1} & 0 & \square & 1 \end{bmatrix} \quad (36)
$$

Besides, by proposition 3.2.7, the image of (36) through the map, for instance, $\phi_{(-1,2),3}$ is the same as the 2-multi-pattern obtained from (35) by replacing each 2-multi-pattern appearing in it by its image by $\phi_{(-1,2),3}$.

4. GENERATION AND RANDOM GENERATION

We exploit now the music box operad to design three random generation algorithms devoted to generate new musical phrases from a finite set of multi-patterns. This relies on colored operads and bud generating systems, a sort of formal grammars introduced in [5].

4.1. Colored operads and bud operads

We provide here the elementary notions about colored operads [12]. We also explain how to build colored operads from operads.

A *set of colors* is any nonempty finite set $\mathfrak{C} := \{\mathsf{b}_1, \dots, \mathsf{b}_k\}$ wherein elements are called *colors*. A $\mathfrak{C}$-*colored set* is a set $\mathcal{C}$ decomposing as a disjoint union

$$
\mathcal{C} := \bigsqcup_{\substack{a \in \mathfrak{C} \\ u \in \mathfrak{C}^*}} \mathcal{C}(a, u) \qquad (37)
$$

where $\mathfrak{C}^*$ is the set of all finite sequences of elements of $\mathfrak{C}$, and the $\mathcal{C}(a, u)$ are sets. For any $x \in \mathcal{C}$, there is by definition a unique pair $(a, u) \in \mathfrak{C} \times \mathfrak{C}^*$ such that $x \in \mathcal{C}(a, u)$. The *arity* $|x|$ of x is the length $|u|$ of u as a word, the *output color* $\mathrm{out}(x)$ of x is a, and for any $i \in [|x|]$, the *i-th input color* $\mathrm{in}_i(x)$ of x is the i-th letter u_i of u. We also denote, for any $n \in \mathbb{N}$, by $\mathcal{C}(n)$ the set of all elements of $\mathcal{C}$ of arity n. Therefore, a colored graded set is in particular a graded set.

A $\mathfrak{C}$-*colored operad* is a triple $(\mathcal{C}, \circ_i, \mathbf{1})$ such that $\mathcal{C}$ is a $\mathfrak{C}$-colored set, $\circ_i$ is a map

$$\underset{i}{\circ} : \mathcal{C}(a, u) \times \mathcal{C}(u_i, v) \to \mathcal{C}\left(a, u \underset{i}{\circ} v\right) \quad i \in [|u|] \quad (38)$$

called *partial composition* map, where $u \circ_i v$ is the word on $\mathfrak{C}$ obtained by replacing the i-th letter of u by v, and $\mathbf{1}$ is a map

$$\mathbf{1} : \mathfrak{C} \to \mathcal{C}(a, a) \quad (39)$$

called *colored unit* map. This data has to satisfy relations (13) and (14) when their left and right members are both well-defined, and, for any $x \in \mathcal{C}$, the relation

$$\mathbf{1}(\mathrm{out}(x)) \underset{1}{\circ} x = x = x \underset{i}{\circ} \mathbf{1}\left(\mathrm{in}_i(x)\right), \; i \in [|x|] \quad (40)$$

Intuitively, an element x of a colored operad having a as output color and u_i as i-th input color for any $i \in [|x|]$ can be seen as an abstract operator wherein colors are assigned to its output and to each of its inputs. Such an operator is depicted as

 (41)

where the colors of the output and inputs are put on the corresponding edges. The partial composition of two elements x and y in a colored operad expresses pictorially as

 (42)

Besides, most of the definitions about operads recalled in section 3.1 generalize straightforwardly to colored operads. In particular, one can consider the full composition map of a colored operad defined by (19) when its right member is well-defined.

Let us introduce another operation, specific to colored operads. Let $(\mathcal{C}, \circ_i, \mathbf{1})$ be a colored operad. The *colored composition* map of $\mathcal{C}$ is the map

$$\odot : \mathcal{C}(a, u) \times \mathcal{C}(b, v) \to \mathcal{C}, \quad a, b \in \mathfrak{C}, \; u, v \in \mathfrak{C}^*, \quad (43)$$

defined, for any $x \in \mathcal{C}(a, u)$ and $y \in \mathcal{C}(b, v)$, by using the full composition map, by

$$x \odot y := x \circ \left[y^{(1)}, \ldots, y^{(|x|)}\right] \quad (44)$$

where for any $i \in [|x|]$,

$$y^{(i)} := \begin{cases} y & \text{if } \mathrm{in}_i(x) = \mathrm{out}(y) \\ \mathbf{1}\left(\mathrm{in}_i(x)\right) & \text{otherwise} \end{cases} \quad (45)$$

Intuitively, $x \odot y$ is obtained by grafting simultaneously the outputs of copies of y into all the inputs of x having the same color as the output color of y.

Let us describe a general construction building a colored operad from a noncolored one introduced in [5]. Given a noncolored operad $(\mathcal{O}, \circ_i, \mathbf{1})$ and a set of colors $\mathfrak{C}$, the $\mathfrak{C}$-*bud operad* of $\mathcal{O}$ is the $\mathfrak{C}$-colored operad $\mathsf{B}_{\mathfrak{C}}(\mathcal{O})$ defined in the following way. First, $\mathsf{B}_{\mathfrak{C}}(\mathcal{O})$ is the $\mathfrak{C}$-colored set defined, for any $a \in \mathfrak{C}$ and $u \in \mathfrak{C}^*$, by

$$\mathsf{B}_{\mathfrak{C}}(\mathcal{O})(a, u) := \{(a, x, u) : x \in \mathcal{O}(|u|)\} \quad (46)$$

Second, the partial composition maps $\circ_i$ of $\mathsf{B}_{\mathfrak{C}}(\mathcal{O})$ are defined, for any $(a, x, u), (u_i, y, v) \in \mathsf{B}_{\mathfrak{C}}(\mathcal{O})$ and $i \in [|u|]$, by

$$(a, x, u) \underset{i}{\circ} (u_i, y, v) := \left(a, x \underset{i}{\circ} y, u \underset{i}{\circ} v\right) \quad (47)$$

where the first occurrence of $\circ_i$ in the right member of (47) is the partial composition map of $\mathcal{O}$ and the second one is a substitution of words : $u \circ_i v$ is the word obtained by replacing in u the i-th letter of u by v. Finally, the colored unit map $\mathbf{1}$ of $\mathsf{B}_{\mathfrak{C}}(\mathcal{O})$ is defined by $\mathbf{1}(a) := (a, \mathbf{1}, a)$ for any $a \in \mathfrak{C}$, where $\mathbf{1}$ is the unit of $\mathcal{O}$. The *pruning* $\mathrm{pr}((a, x, u))$ of an element (a, x, u) of $\mathsf{B}_{\mathfrak{C}}(\mathcal{O})$ is the element x of $\mathcal{O}$.

Intuitively, this construction consists in forming a colored operad $\mathsf{B}_{\mathfrak{C}}(\mathcal{O})$ out of $\mathcal{O}$ by surrounding its elements with an output color and input colors coming from $\mathfrak{C}$ in all possible ways.

We apply this construction to the m-music box operad by setting, for any set $\mathfrak{C}$ of colors,

$$\mathsf{BP}_m^{\mathfrak{C}} := \mathsf{B}_{\mathfrak{C}}\left(\mathsf{P}_m\right) \quad (48)$$

We call $\mathsf{BP}_m^{\mathfrak{C}}$ the $\mathfrak{C}$-*bud m-music box operad*. The elements of $\mathsf{BP}_m^{\mathfrak{C}}$ are called $\mathfrak{C}$-*colored m-multi-patterns*. For instance, for $\mathfrak{C} := \{\mathsf{b}_1, \mathsf{b}_2, \mathsf{b}_3\}$,

$$\left(\mathsf{b}_1, \begin{bmatrix} 1 \,\square\, 0 \,\square\, 1 \\ \overline{7} \,\square\, 0 \; 0 \,\square \end{bmatrix}, \mathsf{b}_2\mathsf{b}_2\mathsf{b}_1\right) \quad (49)$$

is a $\mathfrak{C}$-colored 2-multi-pattern. Moreover, in the colored operad $\mathsf{BP}_2^{\mathfrak{C}}$, one has

$$\left(\mathsf{b}_3, \begin{bmatrix} 0 \; 1 \,\square \\ \overline{1} \,\square\, 0 \end{bmatrix}, \mathsf{b}_2\mathsf{b}_1\right) \underset{2}{\circ} \left(\mathsf{b}_1, \begin{bmatrix} \mathbf{1} \; \mathbf{1} \; \mathbf{2} \\ \mathbf{2} \; \overline{\mathbf{1}} \; \overline{\mathbf{2}} \end{bmatrix}, \mathsf{b}_3\mathsf{b}_3\mathsf{b}_2\right)$$
$$= \left(\mathsf{b}_3, \begin{bmatrix} 0 \; \mathbf{2} \; \mathbf{2} \; \mathbf{3} \,\square \\ \overline{1} \,\square\, \mathbf{2} \; \overline{\mathbf{1}} \; \overline{\mathbf{2}} \end{bmatrix}, \mathsf{b}_2\mathsf{b}_3\mathsf{b}_3\mathsf{b}_2\right) \quad (50)$$

The intuition that justifies the introduction of these colored versions of patterns and of the m-music box operad

is that colors restrict the right to perform the composition of two given patterns. In this way, one can for instance forbid some intervals in the musical phrases specified by the patterns of a suboperad of $\mathrm{BP}_m^{\mathfrak{C}}$ generated by a given set of $\mathfrak{C}$-colored m-multi-patterns. Moreover, given a set $\mathfrak{G}$ of $\mathfrak{C}$-colored m-multi-patterns, the elements of the suboperad $\mathrm{BP}_m^{\mathfrak{C}}{}^{\mathfrak{G}}$ of $\mathrm{BP}_m^{\mathfrak{C}}$ generated by $\mathfrak{G}$ are obtained by composing elements of $\mathfrak{G}$. Therefore, in some sense, these elements inherit from properties of the patterns $\mathfrak{G}$.

The next section uses these ideas to propose random generation algorithms outputting new patterns from existing ones in a controlled way.

4.2. Bud generating systems and random generation

We describe here a sort of generating systems using operads introduced in [5]. Slight variations are considered in this present work. We also design three random generation algorithms to produce musical phrases.

A *bud generating system* [5] is a tuple $(\mathcal{O}, \mathfrak{C}, \mathcal{R}, \mathtt{b})$ where

1. $(\mathcal{O}, \circ_i, \mathbf{1})$ is a noncolored operad, called *ground operad*;
2. $\mathfrak{C}$ is a finite set of colors;
3. $\mathcal{R}$ is a finite subset of $\mathrm{B}_{\mathfrak{C}}(\mathcal{O})$, called *set of rules*;
4. $\mathtt{b}$ is a color of $\mathfrak{C}$, called *initial color*.

For any color $a \in \mathfrak{C}$, we shall denote by $\mathcal{R}_a$ the set of all rules of $\mathcal{R}$ having a as output color.

Bud generating systems are devices similar to context-free formal grammars [8] wherein colors play the role of nonterminal symbols. These last devices are designed to generate sets of words. Bud generating systems are designed to generate more general combinatorial objects (here, m-multi-patterns). More precisely, a bud generating system $(\mathcal{O}, \mathfrak{C}, \mathcal{R}, \mathtt{b})$ allows us to build elements of $\mathcal{O}$ by following three different operating modes. We describe in the next sections the three corresponding random generation algorithms. These algorithms are in particular intended to work with P_m as ground operad in order to generate m-multi-patterns.

Hereafter, we shall provide some examples based upon the bud generating system

$$\mathcal{B} := (\mathrm{P}_2, \{\mathtt{b}_1, \mathtt{b}_2, \mathtt{b}_3\}, \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4, \mathbf{c}_5\}, \mathtt{b}_1) \quad (51)$$

where

$$\mathbf{c}_1 := \left(\mathtt{b}_1, \begin{bmatrix} 0 & 2 & \square & 1 & \square & 0 & 4 \\ \bar{5} & \square & \square & 0 & 0 & 0 & 0 \end{bmatrix}, \mathtt{b}_3 \mathtt{b}_2 \mathtt{b}_1 \mathtt{b}_1 \mathtt{b}_3 \right) \quad (52)$$

$$\mathbf{c}_2 := \left(\mathtt{b}_1, \begin{bmatrix} 1 & \square & 0 \\ 0 & \square & 1 \end{bmatrix}, \mathtt{b}_1 \mathtt{b}_1 \right), \quad \mathbf{c}_3 := \left(\mathtt{b}_2, \begin{bmatrix} \bar{1} \\ 1 \end{bmatrix}, \mathtt{b}_1 \right) \quad (53)$$

$$\mathbf{c}_4 := \left(\mathtt{b}_2, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \mathtt{b}_1 \mathtt{b}_1 \right), \quad \mathbf{c}_5 := \left(\mathtt{b}_3, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathtt{b}_3 \right) \quad (54)$$

Moreover, to interpret the generated multi-patterns, we choose to consider a tempo of 128 and the rooted scale $(\boldsymbol{\lambda}, 9_3)$ where $\boldsymbol{\lambda}$ is the Hirajoshi scale.

Let $\mathcal{B} := (\mathcal{O}, \mathfrak{C}, \mathcal{R}, \mathtt{b})$ be a bud generating system. Let $\rightarrow$ be the binary relation on $\mathrm{B}_{\mathfrak{C}}(\mathcal{O})$ such that

$$(a, x, u) \rightarrow (a, y, v) \quad (55)$$

if there is a rule $r \in \mathcal{R}$ and $i \in [|u|]$ such that

$$(a, y, v) = (a, x, u) \circ_i r \quad (56)$$

An element x of $\mathcal{O}$ is *partially generated* by $\mathcal{B}$ if there is an element $(\mathtt{b}, x, u)$ such that $(\mathtt{b}, \mathbf{1}, \mathtt{b})$ is in relation with $(\mathtt{b}, x, u)$ w.r.t. the reflexive and transitive closure of $\rightarrow$.

For instance, by considering the bud generating system (51), since

$$\left(\mathtt{b}_1, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathtt{b}_1 \right) \rightarrow \left(\mathtt{b}_1, \begin{bmatrix} 1 & \square & 0 \\ 0 & \square & 1 \end{bmatrix}, \mathtt{b}_1 \mathtt{b}_1 \right)$$

$$\rightarrow \left(\mathtt{b}_1, \begin{bmatrix} 1 & \square & 0 & 2 & \square & 1 & \square & 0 & 4 \\ 0 & \square & \bar{4} & \square & \square & 1 & 1 & 1 & 1 \end{bmatrix}, \mathtt{b}_1 \mathtt{b}_3 \mathtt{b}_2 \mathtt{b}_1 \mathtt{b}_1 \mathtt{b}_3 \right) \quad (57)$$

$$\rightarrow \left(\mathtt{b}_1, \begin{bmatrix} 1 & \square & 0 & 2 & 2 & \square & 1 & \square & 0 & 4 \\ 0 & \square & \bar{4} & \square & \square & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \mathtt{b}_1 \mathtt{b}_3 \mathtt{b}_1 \mathtt{b}_1 \mathtt{b}_1 \mathtt{b}_1 \mathtt{b}_3 \right)$$

the 2-multi-pattern

$$\begin{bmatrix} 1 & \square & 0 & 2 & 2 & \square & 1 & \square & 0 & 4 \\ 0 & \square & \bar{4} & \square & \square & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad (58)$$

is partially generated by $\mathcal{B}$.

The *partial random generation algorithm* is the algorithm defined as follows :

— Inputs :
  1. A bud generating system $\mathcal{B} := (\mathcal{O}, \mathfrak{C}, \mathcal{R}, \mathtt{b})$;
  2. An integer $k \geqslant 0$.
— Output : an element of $\mathcal{O}$.
1. Set x as the element $(\mathtt{b}, \mathbf{1}, \mathtt{b})$;
2. Repeat k times :
  (a) Pick a position $i \in [|x|]$ at random;
  (b) If $\mathcal{R}_{\mathrm{in}_i(x)} \neq \emptyset$:
    i. Pick a rule $r \in \mathcal{R}_{\mathrm{in}_i(x)}$ at random;
    ii. Set $x := x \circ_i r$;
3. Returns $\mathrm{pr}(x)$.

This algorithm returns an element partially generated by $\mathcal{B}$ obtained by applying at most k rules to the initial element $(\mathtt{b}, \mathbf{1}, \mathtt{b})$. The execution of the algorithm builds a composition tree of elements of $\mathcal{R}$ with at most k internal nodes.

For instance, by considering the bud generating system (51), this algorithm called with $k := 5$ builds the tree of colored 2-multipatterns

$$(59)$$
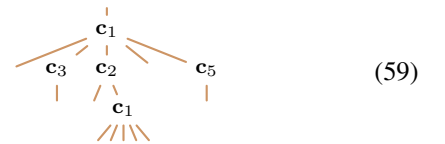
which produces the 2-multi-pattern

$$\begin{bmatrix} 0 & 1 & \square & 2 & \square & 1 & 3 & \square & 2 & \square & 1 & 5 & \square & 0 & 4 \\ \bar{5} & \square & \square & \bar{1} & 0 & \square & \bar{4} & \square & \square & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix} \quad (60)$$

Together with the aforementioned interpretation, the generated musical phrase is

♩ = 128

(musical score)

Let $\rightarrow$ be the binary relation on $\mathsf{B}_{\mathfrak{C}}(\mathcal{O})$ such that

$$(a, x, u) \rightarrow (a, y, v) \qquad (61)$$

if there are rules $r_1, \ldots, r_{|x|} \in \mathcal{R}$ such that

$$(a, y, v) = (a, x, u) \circ [r_1, \ldots, r_{|x|}] \qquad (62)$$

An element x of $\mathcal{O}$ is *fully generated* by $\mathcal{B}$ if there is an element (b, x, u) such that $(\mathsf{b}, \mathbf{1}, \mathsf{b})$ is in relation with (b, x, u) w.r.t. the reflexive and transitive closure of $\rightarrow$.

For instance, by considering the bud generating system (51), since

$$\left(\mathsf{b}_1, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathsf{b}_1 \right)$$

$$\rightarrow \left(\mathsf{b}_1, \begin{bmatrix} 0\ 2\ \square\ 1\ \square\ 0\ 4 \\ \bar{5}\ \square\square\ 0\ 0\ 0\ 0 \end{bmatrix}, \mathsf{b}_3\mathsf{b}_2\mathsf{b}_1\mathsf{b}_1\mathsf{b}_3 \right)$$

$$\rightarrow \left(\mathsf{b}_1, \begin{bmatrix} 0\ 1\ \square\ 2\ \square\ 1\ \square\ 0\ 2\ \square\ 1\ \square\ 0\ 4\ 4 \\ \bar{5}\ \square\square\ \bar{1}\ 0\ \square\ 1\ 5\ \square\square\ 0\ 0\ 0\ 0\ 0 \end{bmatrix}, \right.$$

$$\left. \mathsf{b}_3\mathsf{b}_1\mathsf{b}_1\mathsf{b}_1\mathsf{b}_3\mathsf{b}_2\mathsf{b}_1\mathsf{b}_1\mathsf{b}_3\mathsf{b}_3 \right) \qquad (63)$$

the 2-multi-pattern

$$\begin{bmatrix} 0\ 1\ \square\ 2\ \square\ 1\ \square\ 0\ 2\ \square\ 1\ \square\ 0\ 4\ 4 \\ \bar{5}\ \square\square\ \bar{1}\ 0\ \square\ 1\ 5\ \square\square\ 0\ 0\ 0\ 0\ 0 \end{bmatrix} \qquad (64)$$

is fully generated by $\mathcal{B}$.

The *full random generation algorithm* is the algorithm defined as follows :
— Inputs :
    1. A bud generating system $\mathcal{B} := (\mathcal{O}, \mathfrak{C}, \mathcal{R}, \mathsf{b})$;
    2. An integer $k \geqslant 0$.
— Output : an element of $\mathcal{O}$.
1. Set x as the element $(\mathsf{b}, \mathbf{1}, \mathsf{b})$;
2. Repeat k times :
    (a) If all $\mathcal{R}_{\mathrm{in}_i(x)}$, $i \in [|x|]$, are nonempty :
        i. Let $(r_1, \ldots, r_{|x|})$ be a tuple of rules such that each r_i is picked at random in $\mathcal{R}_{\mathrm{in}_i(x)}$;
        ii. Set $x := x \circ [r_1, \ldots, r_{|x|}]$;
3. Return $\mathrm{pr}(x)$.

This algorithm returns an element synchronously generated by $\mathcal{B}$ obtained by applying at most k rules to the initial element $(\mathsf{b}, \mathbf{1}, \mathsf{b})$. The execution of the algorithm builds a composition tree of elements of $\mathcal{R}$ of height at most $k + 1$ wherein the leaves are all at the same distance from the root.

For instance, by considering the bud generating system (51), this algorithm called with $k := 2$ builds the tree of colored 2-multipatterns

(tree diagram)

$$(65)$$
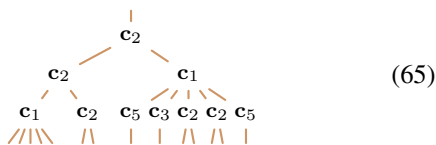
which produces the 2-multi-pattern

$$\begin{bmatrix} 2\ 4\ \square\ 3\ \square\ 2\ 6\ \square\ 2\ \square\ 1\ \square\ 0\ 1\ \square\ 2\ \square\ 1\ \square\ 1\ \square\ 0\ 4 \\ \bar{5}\ \square\square\ 0\ 0\ 0\ 0\ \square\ 1\ \square\ 2\ \square\ \bar{4}\ \square\square\ 0\ 1\ \square\ 2\ 1\ \square\ 2\ 1 \end{bmatrix} \qquad (66)$$

Together with the aforementioned interpretation, the generated musical phrase is

♩ = 128

(musical score)

Let $\xrightarrow{\odot}$ be the binary relation on $\mathsf{B}_{\mathfrak{C}}(\mathcal{O})$ such that

$$(a, x, u) \xrightarrow{\odot} (a, y, v) \qquad (67)$$

if there is a rule $r \in \mathcal{R}$ such that

$$(a, y, v) = (a, x, u) \odot r \qquad (68)$$

An element x of $\mathcal{O}$ is *colorfully generated* by $\mathcal{B}$ if there is an element (b, x, u) such that $(\mathsf{b}, \mathbf{1}, \mathsf{b})$ is in relation with (b, x, u) w.r.t. the reflexive and transitive closure of $\xrightarrow{\odot}$.

For instance, by considering the bud generating system (51), since

$$\left(\mathsf{b}_1, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathsf{b}_1 \right) \xrightarrow{\odot} \left(\mathsf{b}_1, \begin{bmatrix} 0\ 2\ \square\ 1\ \square\ 0\ 4 \\ \bar{5}\ \square\square\ 0\ 0\ 0\ 0 \end{bmatrix}, \mathsf{b}_3\mathsf{b}_2\mathsf{b}_1\mathsf{b}_1\mathsf{b}_3 \right)$$

$$\xrightarrow{\odot} \left(\mathsf{b}_1, \begin{bmatrix} 0\ 2\ \square\ 2\ \square\ 1\ \square\ 1\ \square\ 0\ 4 \\ \bar{5}\ \square\square\ 0\ 0\ \square\ 1\ 0\ \square\ 1\ 0 \end{bmatrix}, \mathsf{b}_3\mathsf{b}_2\mathsf{b}_1\mathsf{b}_1\mathsf{b}_1\mathsf{b}_3 \right)$$

$$\xrightarrow{\odot} \left(\mathsf{b}_1, \begin{bmatrix} 0\ 1\ \square\ 2\ \square\ 1\ \square\ 1\ \square\ 0\ 4 \\ \bar{5}\ \square\square\ \bar{1}\ 0\ \square\ 1\ 0\ \square\ 1\ 0 \end{bmatrix}, \mathsf{b}_3\mathsf{b}_1\mathsf{b}_1\mathsf{b}_1\mathsf{b}_1\mathsf{b}_3 \right) \qquad (69)$$

the 2-multi-pattern

$$\begin{bmatrix} 0\ 1\ \square\ 2\ \square\ 1\ \square\ 1\ \square\ 0\ 4 \\ \bar{5}\ \square\square\ \bar{1}\ 0\ \square\ 1\ 0\ \square\ 1\ 0 \end{bmatrix} \qquad (70)$$

is colorfully generated by $\mathcal{B}$.

The *colored random generation algorithm* is the algorithm defined as follows :
— Inputs :
    1. A bud generating system $\mathcal{B} := (\mathcal{O}, \mathfrak{C}, \mathcal{R}, \mathsf{b})$;
    2. An integer $k \geqslant 0$.
— Output : an element of $\mathcal{O}$.
1. Set x as the element $(\mathsf{b}, \mathbf{1}, \mathsf{b})$;
2. Repeat k times :
    (a) Pick a rule $r \in \mathcal{R}$ at random ;
    (b) Set $x := x \odot r$;
3. Returns $\mathrm{pr}(x)$.

This algorithm returns an element colorfully generated by $\mathcal{B}$ obtained by applying at most k rules to the initial element $(\mathsf{b}, \mathbf{1}, \mathsf{b})$. The execution of the algorithm builds a composition tree of elements of height at most $k + 1$.

For instance, by considering the bud generating system (51), this algorithm called with $k := 3$ builds the tree

of colored 2-multipatterns

$$(71)$$
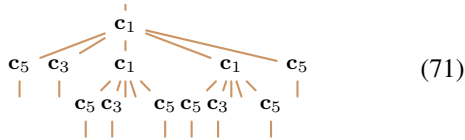
which produces the 2-multi-pattern

$$\begin{bmatrix} 0\ 1\ \square\ 1\ 2\ \square\ 2\ \square\ 1\ 5\ \square\ 0\ 1\ \square\ 1\ \square\ 0\ 4\ 4 \\ 5\ \square\ \square\ \bar{1}\ 5\ \square\ \square\ \bar{1}\ 0\ 0\ 0\ \bar{5}\ \square\ \square\ \bar{1}\ 0\ 0\ 0\ 0 \end{bmatrix} \quad (72)$$

Together with the aforementioned interpretation, the generated musical phrase is

5. APPLICATIONS : EXPLORING VARIATIONS OF PATTERNS

We construct here some particular bud generated systems devoted to work with the algorithms introduced in Section 4.2. They generate variations of a single 1-multi-pattern $\mathbf{p}$ given at input, with possibly some auxiliary data. Each performs a precise musical transformation of $\mathbf{p}$.

5.1. Random temporizations

Given a pattern $\mathbf{p}$ and an integer $t \geqslant 1$, we define the *temporizator bud generating system* $\mathcal{B}_{\mathbf{p},t}^{\mathrm{tem}}$ of $\mathbf{p}$ and t by

$$\mathcal{B}_{\mathbf{p},t}^{\mathrm{tem}} := (\mathsf{P}_1, \mathfrak{C}, \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_1', \dots, \mathbf{c}_t'\}, \mathsf{b}_1) \quad (73)$$

where $\mathfrak{C}$ is the set of colors $\{\mathsf{b}_1, \mathsf{b}_2, \mathsf{b}_3\}$ and $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_1', \dots,$ $\mathbf{c}_t'$ are the $\mathfrak{C}$-colored 1-multi-patterns

$$\mathbf{c}_1 := \left(\mathsf{b}_1, \mathbf{p}, \mathsf{b}_2^{|\mathbf{p}|}\right), \quad \mathbf{c}_2 := \left(\mathsf{b}_2, \mathbf{p}, \mathsf{b}_2^{|\mathbf{p}|}\right) \quad (74\mathrm{a})$$

$$\mathbf{c}_j' := \left(\mathsf{b}_2, \left[0\ \square^j\right], \mathsf{b}_3\right), \quad j \in [t]. \quad (74\mathrm{b})$$

The temporizator bud generating system of $\mathbf{p}$ and t generates a version of the pattern $\mathbf{p}$ composed with itself where the durations of some beats have been increased by at most t. The colors, and in particular the color b_3, prevent multiple compositions of the colored patterns $\mathbf{c}_j'$, $j \in [t]$, in order to not overly increase the duration of some beats.

For instance, by considering the pattern $\mathbf{p} := 02\square1\square04$ and the parameter $t := 2$, the partial random generation algorithm, ran with the bud generating system $\mathcal{B}_{\mathbf{p},t}^{\mathrm{tem}}$ and $k := 16$ as inputs, produces the pattern

$$02\square\square\square1\square3\square\square\square2\square15\square0\square\square4\square \quad (75)$$

Together with the interpretation consisting in a tempo of 128 and the rooted scale $(\boldsymbol{\lambda}, 9_3)$ where $\boldsymbol{\lambda}$ is the Hirajoshi scale, the generated musical phrase is

5.2. Random rhythmic variations

Given a pattern $\mathbf{p}$ and a rhythm pattern $\mathbf{r}$, we define the *rhythmic bud generating system* $\mathcal{B}_{\mathbf{p},\mathbf{r}}^{\mathrm{rhy}}$ of $\mathbf{p}$ and $\mathbf{r}$ by

$$\mathcal{B}_{\mathbf{p},\mathbf{r}}^{\mathrm{rhy}} := (\mathsf{P}_m, \mathfrak{C}, \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}, \mathsf{b}_1) \quad (76)$$

where $\mathfrak{C}$ is the set of colors $\{\mathsf{b}_1, \mathsf{b}_2, \mathsf{b}_3\}$ and $\mathbf{c}_1, \mathbf{c}_2$, and $\mathbf{c}_3$ are the three $\mathfrak{C}$-colored 1-multi-patterns

$$\mathbf{c}_1 := \left(\mathsf{b}_1, \mathbf{p}, \mathsf{b}_2^{|\mathbf{p}|}\right), \quad \mathbf{c}_2 := \left(\mathsf{b}_2, \mathbf{p}, \mathsf{b}_2^{|\mathbf{p}|}\right) \quad (77\mathrm{a})$$

$$\mathbf{c}_3 := \left(\mathsf{b}_2, \mathbf{r}', \mathsf{b}_3^{|\mathbf{r}|}\right) \quad (77\mathrm{b})$$

where $\mathbf{r}'$ is the pattern $\left(0^{|\mathbf{r}|}, \mathbf{r}\right)$. The rhythmic bud generating system of $\mathbf{p}$ and $\mathbf{r}$ generates a version of the pattern $\mathbf{p}$ composed with itself where some beats are repeated accordingly to the rhythm pattern $\mathbf{r}$. The colors, and in particular the color b_3, prevent multiple compositions of the colored pattern $\mathbf{c}_3$. Observe that when $\mathbf{r} = \epsilon$, each composition involving $\mathbf{c}_3$ deletes a beat in the generated pattern.

For instance, by considering the pattern $\mathbf{p} := 1\square011\square2$ and the rhythm pattern $\mathbf{r} := \blacksquare\blacksquare\square\blacksquare\square$, the partial random generation algorithm run with the bud generating system $\mathcal{B}_{\mathbf{p},\mathbf{r}}^{\mathrm{rhy}}$ and $k := 8$ as inputs produces the pattern

$$22\square\square2\square\square122\square3\square1\square011\square22\square122\square\square2\square2\square31\square22\square\square2\square \quad (78)$$

Together with the interpretation consisting in a tempo of 128 and the rooted scale $(\boldsymbol{\lambda}, 9_3)$ where $\boldsymbol{\lambda}$ is the minor natural scale, the generated musical phrase is

5.3. Random harmonizations

For any pattern $\mathbf{p}$ and an integer $m \geqslant 1$, we denote by $[\mathbf{p}]_m$ the m-multi-pattern $\left([\mathbf{p}]_m^{(1)}, \dots, [\mathbf{p}]_m^{(m)}\right)$ satisfying $[\mathbf{p}]_m^{(i)} = \mathbf{p}$ for all $i \in [m]$.

Given a pattern $\mathbf{p}$ and a degree pattern $\mathbf{d}$ of arity $m \geqslant 1$, we define the *harmonizator bud generating system* $\mathcal{B}_{\mathbf{p},\mathbf{d}}^{\mathrm{har}}$ of $\mathbf{p}$ and $\mathbf{d}$ by

$$\mathcal{B}_{\mathbf{p},\mathbf{d}}^{\mathrm{arp}} := (\mathsf{P}_m, \mathfrak{C}, \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}, \mathsf{b}_1) \quad (79)$$

where $\mathfrak{C}$ is the set of colors $\{\mathsf{b}_1, \mathsf{b}_2, \mathsf{b}_3\}$ and $\mathbf{c}_1, \mathbf{c}_2$, and $\mathbf{c}_3$ are the three $\mathfrak{C}$-colored m-multi-patterns

$$\mathbf{c}_1 := (\mathsf{b}_1, [\mathbf{p}]_m, \mathsf{b}_2^m), \quad \mathbf{c}_2 := (\mathsf{b}_2, [\mathbf{p}]_m, \mathsf{b}_2^m), \quad (80\mathrm{a})$$

$$\mathbf{c}_3 := (\mathsf{b}_2, [\mathbf{d}]_m, \mathsf{b}_3) \quad (80\mathrm{b})$$

The harmonizator bud generating system of $\mathbf{p}$ and $\mathbf{d}$ generates an harmonized version of the pattern $\mathbf{p}$ composed with itself, with chords controlled by $\mathbf{d}$. The colors, and in particular the color b_3, prevent multiple compositions of the colored pattern $\mathbf{c}_3$.

For instance, by considering the degree pattern $\mathbf{d} := 05\bar{7}$ and the pattern $\mathbf{p} := 2102\square1\square0\square$, the partial random

generation algorithm run with the bud generating system $\mathcal{B}_{\mathbf{p},\mathbf{d}}^{\mathrm{har}}$ and $k := 3$ as inputs produces the 3-multi-pattern

$$\begin{bmatrix} 2\,1\,0\,2\,\square\,1\,\square\,0\,\square \\ 2\,6\,5\,2\,\square\,1\,\square\,0\,\square \\ 2\,\bar{6}\,\bar{7}\,2\,\square\,1\,\square\,0\,\square \end{bmatrix} \tag{81}$$

Together with the interpretation consisting in a tempo of 128 and the rooted scale $(\boldsymbol{\lambda}, 9_3)$ where $\boldsymbol{\lambda}$ is the minor natural scale, the generated musical phrase is

5.4. Random arpeggiations

From a given pattern $\mathbf{p}$ and a degree pattern $\mathbf{d}$ of arity $m \geqslant 1$, we define the *arpeggiator bud generating system* $\mathcal{B}_{\mathbf{p},\mathbf{d}}^{\mathrm{arp}}$ of $\mathbf{p}$ and $\mathbf{d}$ by

$$\mathcal{B}_{\mathbf{p},\mathbf{d}}^{\mathrm{arp}} := (\mathsf{P}_m, \mathfrak{C}, \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}, \mathsf{b}_1) \tag{82}$$

where $\mathfrak{C}$ is the set of colors $\{\mathsf{b}_1, \mathsf{b}_2, \mathsf{b}_3\}$ and $\mathbf{c}_1$, $\mathbf{c}_2$, and $\mathbf{c}_3$ are the three $\mathfrak{C}$-colored m-multipatterns

$$\mathbf{c}_1 := (\mathsf{b}_1, [\mathbf{p}]_m, \mathsf{b}_2^m), \quad \mathbf{c}_2 := (\mathsf{b}_2, [\mathbf{p}]_m, \mathsf{b}_2^m), \tag{83a}$$

$$\mathbf{c}_3 := \left(\mathsf{b}_2, \begin{bmatrix} \mathbf{d}_1 & \square & \square & \dots & \square \\ \square & \mathbf{d}_2 & \square & \dots & \square \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \square & \square & \dots & \square & \mathbf{d}_m \end{bmatrix}, \mathsf{b}_3 \right) \tag{83b}$$

The arpeggiator bud generating system of $\mathbf{p}$ and $\mathbf{d}$ generates an arpeggiated version of the pattern $\mathbf{p}$ composed with itself, where the arpeggio is controlled by $\mathbf{d}$. The colors, and in particular the color b_3, prevent multiple compositions of the colored pattern $\mathbf{c}_3$. Observe in particular that when $\mathbf{d} = 0^m$, each composition involving $\mathbf{c}_3$ creates a repetition of a same beat over the m stacked voices.

For instance, by considering the pattern $\mathbf{p} := 0\square213\square1$ and the degree pattern $\mathbf{d} := 024$, the partial random generation algorithm run with the bud generating system $\mathcal{B}_{\mathbf{p},\mathbf{d}}^{\mathrm{arp}}$ and $k := 8$ as inputs produces the 3-multi-pattern

$$\begin{bmatrix} 0\,\square\,2\,1\,\square\,\square\,3\,\square\,1\,\square\,2\,\square\,4\,3\,5\,\square\,3\,\square\,\square\,1\,\square\,\square\,3\,\square\,1 \\ 0\,\square\,2\,\square\,3\,\square\,3\,\square\,1\,\square\,2\,\square\,4\,3\,5\,\square\,\square\,5\,\square\,\square\,3\,\square\,3\,\square\,1 \\ 0\,\square\,2\,\square\,\square\,5\,3\,\square\,1\,\square\,2\,\square\,4\,3\,5\,\square\,\square\,\square\,7\,\square\,\square\,5\,3\,\square\,1 \end{bmatrix} \tag{84}$$

Together with the interpretation consisting in a tempo of 128 and the rooted scale $(\boldsymbol{\lambda}, 0_3)$ where $\boldsymbol{\lambda}$ is the major natural scale, the generated musical phrase is

6. RÉFÉRENCES

[1] Andreatta, M. « From music to mathematics and backwards: introducing algebra, topology and category theory into computational musicology », *Imagine Math* 6 (2018), p. 77-88.

[2] Briot, J.-P., Hadjeres, G., Pachet, F.-D. *Deep Learning Techniques for Music Generation*. Springer, Cham, 2020.

[3] Giraudo, S. Combinatorial operads from monoids, *Journal of Algebraic Combinatorics* 41/2 (2015), p. 493-538.

[4] Giraudo, S. *Nonsymmetric Operads in Combinatorics*. Springer, Cham, 2018.

[5] Giraudo, S. Colored operads, series on colored operads, and combinatorial generating systems, *Discrete Mathematics* 342/6 (2019), p. 1624-1657.

[6] Giraudo, S. Bud Music Box. *GitHub repository*, 2019-2020. `github.com/SamueleGiraudo/Bud-Music-Box`, accédé le 16 octobre 2020.

[7] Holtzman, S. R. Using Generative Grammars for Music Composition. *Computer Music Journal* 5/1 (1981), p. 51-64.

[8] Hopcroft, J. E., Matwani, R., Ullman, J. D. *Introduction to Automata Theory, Languages, and Computation*. Pearson, London, 3rd edition, 2006.

[9] Jedrzejewski, F. *Hiérotopies musicales : Modèles mathématiques de la musique*. Hermann, Paris, 2019.

[10] Matić, D. A genetic algorithm for composing music. *Yugoslavian Journal Operations Research* 20/1 (2010), p.157-177, 2010.

[11] Méndez, M. A. *Set operads in combinatorics and computer science*. Springer, Cham, 2015.

[12] Yau, D. *Colored Operads*. American Mathematical Society, Providence, 2016.

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